Chordality for Simplicial Complexes

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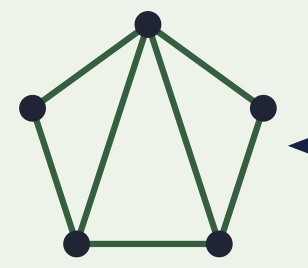
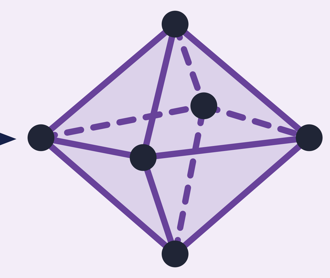
A graph G



A d-dimensional simplicial complex Δ



Paper

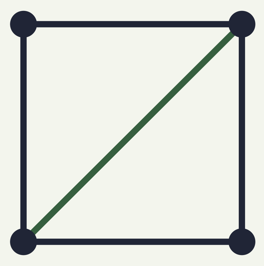
Website

A graph is exactly a 1-dimensional simplicial complex.

For graphs, the following are equivalent

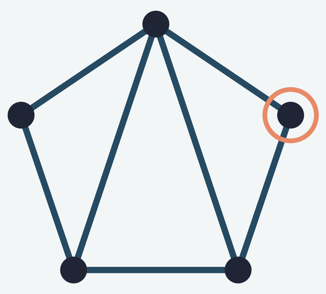
1. Induced cycles

No induced cycles of length ≥ 4.



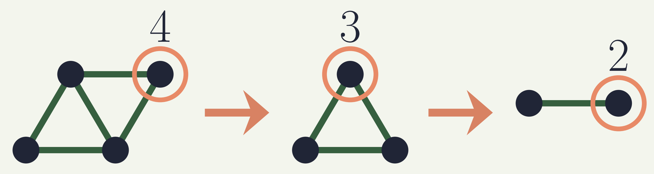
2. Simplicial vertices

Every nonempty induced subgraph has a simplicial vertex.



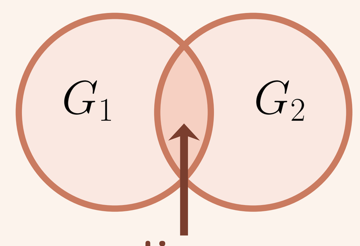
3. Perfect elimination order

There is an ordering $v_1, \dots, v_n$ such that each v_i is simplicial in the graph remaining after $v_1, \dots, v_{i-1}$ are deleted.



4. Clique decomposition

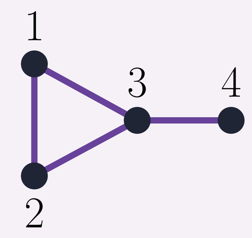
$G = G_1 \cup G_2$, where G_1, G_2 are proper induced chordal subgraphs and $G_1 \cap G_2$ is a clique.



5. Alexander dual NEW

$G^\vee$ is vertex decomposable.

Known by Fröberg ('90), Eagon–Reiner ('98): $\mathrm{Cl}(G)^\vee$ is vertex decomposable.



$G^\vee = \langle 13, 23, 4 \rangle$
$\mathrm{Cl}(G) = \langle 123, 34 \rangle$
$\mathrm{Cl}(G)^\vee = \langle 13, 23 \rangle$

☆ Can the rich package of equivalent characterizations of chordal graphs be generalized to d-dimensional simplicial complexes?

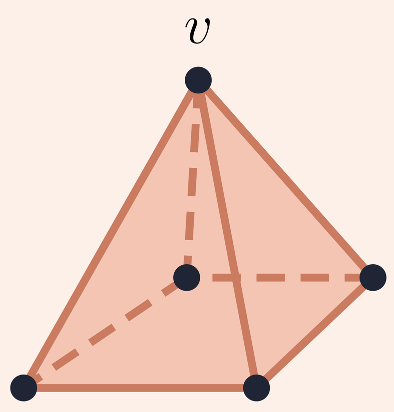
Existing notions of higher-dimensional chordalities

E-chordality: Emtander (2010) — vertex-order viewpoint

W-chordality: Woodroofe (2011) — simplicial-vertex viewpoint

Ridge-chordality: Bigdeli et al. (2017) — codimension-one elimination viewpoint

What fails?



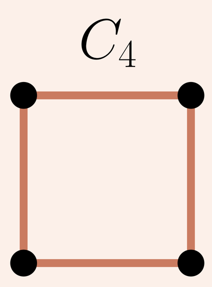
$\Delta = v * C_4$

Δ is ridge-chordal, but

$\mathrm{del}(v, \Delta) = C_4$ and

$\mathrm{link}(v, \Delta) = C_4$ and

$\mathrm{skel}_1(\Delta)$ are not chordal.



Why strengthen instead of weaken?

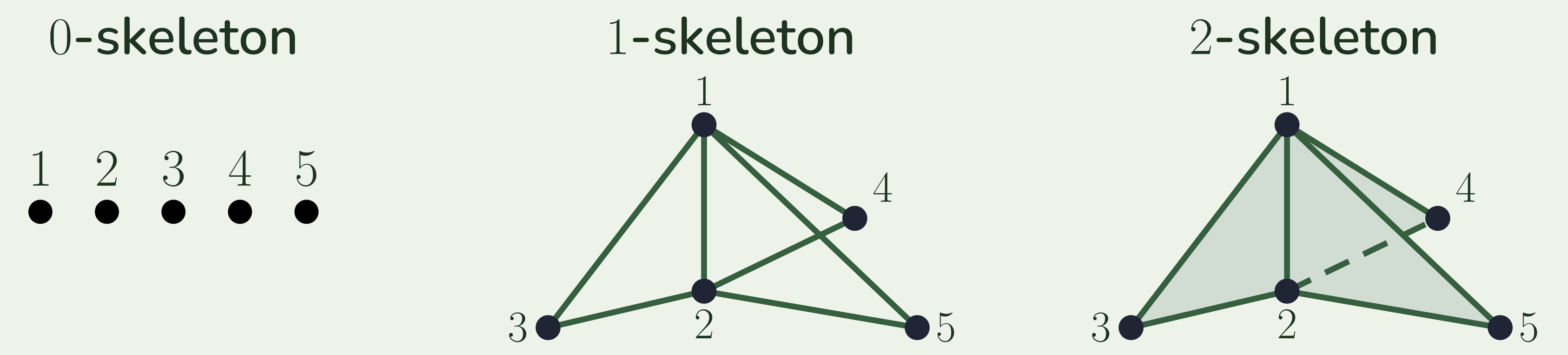
Previous approaches mainly weakened chordality to recover individual classical characterizations.

Strengthening instead gives:

one vertex order → all skeleta → stability under deletions, links, and skeleta → induction → several classical characterizations reappear

Skeleton-E-chordality NEW

Δ is **skeleton-E-chordal** if it admits a vertex labeling such that whenever distinct faces $f, g \in \Delta$ have the same cardinality and the same maximum vertex, every face $h \subseteq f \cup g$ with that same cardinality also lies in Δ.



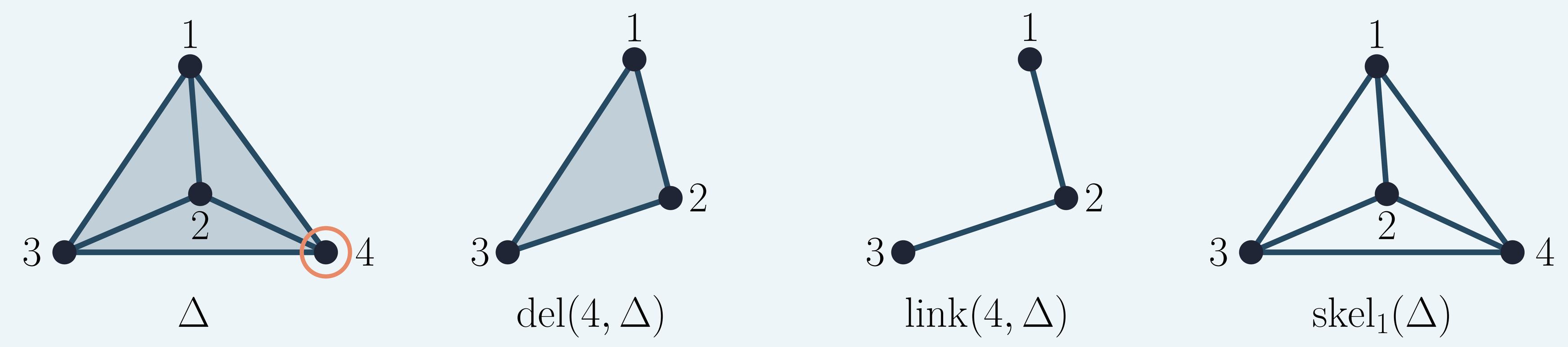
Equivalently, every skeleton is E-chordal with one common labeling.

Skeleton-clique-chordality NEW

Δ is **skeleton-clique-chordal** $\iff$ $\mathrm{skel}_1(\Delta)$ is a chordal graph.

Hereditary properties NEW

Skeleton-E-chordality is closed under deletions, links, and skeleta.



Vertex decomposability

A vertex v is **shedding** if no facet of $\mathrm{link}(v, \Delta)$ is a facet of $\mathrm{del}(v, \Delta)$.

A complex is **vertex decomposable** if it is a simplex, or it has a shedding vertex v such that both $\mathrm{del}(v, \Delta)$ and $\mathrm{link}(v, \Delta)$ are vertex decomposable.

Alexander dual

$\Delta^\vee = \{F \subseteq V : V \setminus F \notin \Delta\}$.

Subflag complexes

Δ is **subflag** $\iff$ it has no missing faces of dimensions $2, \dots, \dim \Delta$.

What lifts to higher dimensions?

A. Simplicial vertices NEW

Δ is skeleton-E-chordal

$\iff$

every nonempty induced subcomplex of Δ has a skeleton-E-simplicial vertex.

B. Clique decomposition NEW

For subflag complexes:

Δ is skeleton-E-chordal

$\iff$

it splits as $\Delta = \Delta_1 \cup \Delta_2$, with each Δ_i a skeleton-E-chordal induced subcomplex of Δ, and with $\Delta_1 \cap \Delta_2$ a complex whose 1-skeleton is a clique.

C. Alexander dual NEW

Δ is skeleton-E-chordal

$\Longrightarrow$

$\Delta^\vee$ is vertex decomposable

$\Longrightarrow$

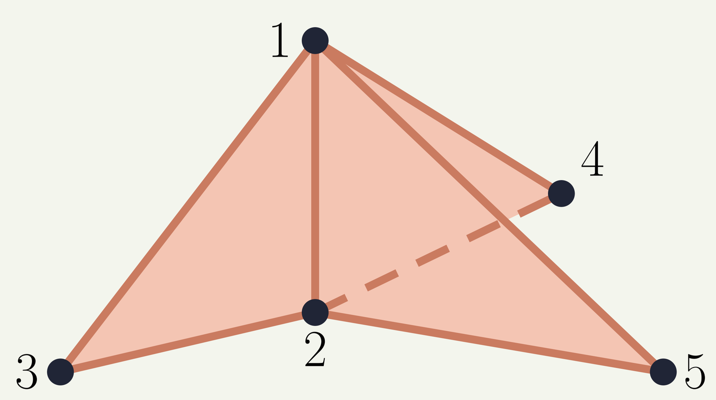
Δ is skeleton-clique-chordal.

If Δ is subflag, all three properties are equivalent.

A. Simplicial vertices

Vertex 5 is skeleton-E-simplicial.

In the 1-skeleton, edges 15 and 25 force 12.



B. Clique decomposition

$\Delta = \Delta_1 \cup \Delta_2$,
$\Delta_1 = \langle 123, 124 \rangle$,
$\Delta_2 = \langle 125 \rangle$,
$\Delta_1 \cap \Delta_2 = \langle 12 \rangle$.

C. Alexander dual

$\Delta^\vee = \Delta$ (self-dual).

Shedding vertices: 5, then 4.

By strengthening chordality across all skeleta, several classical characterizations of chordal graphs reappear for simplicial complexes.